Numerical simulation of spaceborn VLF spectrograms related to lightning-induced emissions

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We consider the key problems in the applications of geometrical optics (GO) to numerical simulations of VLF spectrograms related to lightning-induced emissions. The following points are discussed: a) wave field expansion in an inhomogeneous medium into GO wave packets; b) construction of frequency-time plots on spectrograms, i.e. finite-width curves on which the spectral intensity differs from zero; and c) determining the spectral amplitude from the wave packet amplitude, with the account of the evolution of the latter in space and time. The application of the discussed ideas results in a consistent approach to spectrogram modeling implemented in the form of program package that permits to construct both the detailed (short time scale) spectrograms given the satellite position and the properties of lightning source, and overview (large time scale) spectrograms corresponding to the satellite trajectory and lightning statistics.

1. Introduction
Analysis of spectrograms has been the main tool in whistler studies since classical work by Storey [1953] devoted to ground-based measurements. The variety of whistler spectrograms observed on the ground was essentially enlarged with the beginning of satellite observations. In particular, in the spectrograms of wave data from OGO 1 and 3, Smith and Angerami [1968] found, and basically explained, a new type of whistlers, the so-called magnetospherically reflected (MR) whistlers, virtually predicted by Kimura [1966]. An important contribution to MR whistler studies was made by Edgar [1976], who pointed out (among other things) that the frequency-time plot on MR whistler spectrogram may be explained, in a general way, on the basis of ray tracing.

Recently, while the interest to satellite VLF observations has been stimulated by extensive data from MAGION 4 and 5 (Jiříček and Tříška, [1998]), numerical modeling of VLF spectrograms related to lightning-induced emissions has become a new line in the magnetospheric whistler studies (Shklyar and Jiříček, [2000]; Lundin and Krafft, [2001]; Borůvka et al., [2003]). However, most of work in this direction was based on intuitive ideas, but not on well grounded concepts. The main goal of this report is to fill up this gap. The points to be explicated are those listed in the Abstract.

2. Basic steps in spectrogram modeling
2.1 Expansion of the wave field into wave packets
Applications of ray tracing to spectrogram modeling require that the wave field is represented in the form of GO wave packets. (For the sake of shortness, the consideration below is limited to one-dimensional case). In a homogeneous medium, the expansion of initial field can be obtained from standard Fourier transformation by dividing the axis $k$ into intervals of the length $2\delta k$ centered on $k_n$, and writing the integral over $k$ as the sum of integrals over these intervals, which results in the expansion

$$f(x) = \sum_n F_n(x)e^{ik_nx}$$

where $F_n(x) = \int_{-\infty}^{\infty} f(x')e^{-ik_nx'}dx'$.

Assuming that $f(x)$ represents the initial field, we may write the expression for the field at $t > 0$ in the required form

$$f(x,t) = \sum_n \Phi_n(x,t)e^{ik_nx - i\omega(k_n)t},$$

where $\omega(k_n)$ are determined by the dispersion relation, while the functions $\Phi_n(x,t)$ satisfy the initial conditions $\Phi_n(x,t=0) = F_n(x)$ and are slowly varying functions of both $x$ and $t$.

In an inhomogeneous medium, the dispersion relation $\omega = \omega(k,x)$ defines the wave number $k$ as the function of $\omega$ and $x$, and the eikonal

$$\Psi(x,t;\omega) = \psi(x,\omega) - \omega t; \quad \psi(x,\omega) \equiv \int_0^x k(x',\omega) dx',$$

(1)

so that, for a given $\omega$, the wave packet $\Phi(x,t;\omega)e^{i\Psi(x,t;\omega)}$ with a slowly varying amplitude $\Phi(x,t;\omega)$ is a solution of the wave equation. Accordingly, we look for the expansion of the field in the form

$$\sum_n \Phi_n(x,t)e^{i\Psi_n(x,t)}; \quad \Psi_n(x,t) = \Psi(x,t,\omega_n),$$

where $\Psi(x,t;\omega)$ is defined in (1). If the main contribution to the expansion comes from a small region of $\omega$ close to $\omega_0$, the problem can be reduced to standard Fourier expansion, as in the homogeneous case, and results in the following presentation of the wave field.
\[ f(x, t) = \sum_{n} \Phi_n(x, t) e^{i\Psi_n(x, t)}; \quad \Phi_n^0(x) = \int_{-\infty}^{\infty} f^0(x') e^{-i\Psi_n^0(x')} \sin\left\{ \frac{\delta \lambda}{\pi} [s(x) - s(x')] \right\} ds(x') dx', \]

where the superscript “0” denotes the initial values, and the following notation has been introduced

\[ s(x) = \frac{\partial \psi(x, \omega)}{\partial \omega} \bigg|_{\omega=\omega_0}; \quad \lambda = (\omega - \omega_0). \]

2.2 Constructing frequency-time plot
In contrast to the method used by Bortnik et al. [2003], which employs the notion of a detection area, we use the method suggested by Storey (private communication) based on the concept of group fronts. Detailed discussion and comparison of these two ways of constructing the frequency-time plot on spectrograms is given in Shklyar et al., [2004].

2.3 Spectral intensity versus wave packet amplitude
When the wave field in the given frequency band is that of one wave packet, the corresponding spectral intensity displayed on spectrogram may be expressed through the wave packet amplitude with the help of Parseval’s relation and the energy conservation. Finding the wave packet amplitude from the wave energy density requires the calculation of the wave packet parameters, including the variation of the ray tube cross section, along the ray trajectory.

3. Examples of real and model spectrograms
(See Figures to the right).

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References
Shklyar, D. R., and Jiřiček, F., Characteristic properties of Nu whistlers as inferred from observations and numerical modeling, Annales Geophysicae, 22, 1–18, 2004.

Fig. 1. Magnetospherically reflected (MR) whistler spectrogram observed by the MAGION 5 satellite on orbit 4224 (top panel) and simulated spectrogram of MR whistler reproducing the main features of the real one (bottom panel).

Fig. 2. Overview spectrogram along MAGION 5 orbit 7102 (top panel) and its simulated counterpart (bottom panel).