Relativistic Thermodynamics with Angular Momentum

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The equilibrium state of an isolated system is characterized by its conserved quantities. The total energy and particle number are usually taken into account in conventional thermodynamics, however, angular momentum is often overlooked. In the present study, relativistic thermodynamics has been generalized to accommodate angular momentum as a thermodynamical parameter. The resulting relativistic equilibrium has a remarkable difference from the one without angular momentum: a rotating body has a global temperature that is different from the local temperatures. Blackbody radiation from a rotating body is calculated as an application of this global temperature.

1. Relativistic Thermodynamics

Attempts to generalize thermodynamics to accommodate relativity have been made right after the establishment of the relativistic physics (Einstein 1907, Plank, 1908). In 1960s, there had been a heated controversy on the covariant definition of thermodynamical values such as temperature or heat. A number of different theories have been proposed; for example, the paper by Lansberg & Johns (1967) has a table of twenty different definitions of thermodynamical values. The discussion came to a general agreement that there can be several different theories, each of which has self-consistent definitions of thermodynamical quantities within its framework (Yuen, 1970). Having accepted that, the choice of the theory can be based on aesthetics or convenience. There are authors who prefer the formulation proposed by van Kampen (1967) on this basis (Yuen, 1970; Israel, 1976).

The authors of the present paper also favorable to the van Kampen’s formulation, where the momentum is regarded as an extensive thermodynamical quantity like energy. The energy-momentum four vector is treated as a thermodynamical parameter. The resulting relativistic equilibrium should be uniquely determined with a small number of global quantities (energy, temperature, etc.). To overcome this difficulty we can define a global temperature when the body is in equilibrium. The entropy in Equation (1) is then expressed as

\[
dS = \beta \mu dP^\mu + \lambda_{\mu \nu} dL^{\mu \nu},
\]

where \(P^\mu\) is the energy-momentum four vector and \(\beta\) is the inverse temperature four vector defined as \(\beta = \beta u_\mu (\beta: \text{inverse temperature}, u_\mu: \text{four velocity})\).

Provided momentum can be treated as an extensive thermodynamical quantity, the same treatment should be valid for another kind of extensive quantity, namely, angular momentum. In non-relativistic thermodynamics, it is rather a trivial generalization to include angular momentum as a thermodynamical quantity (Landau, 1958). However, relativity gives rise to problems that cannot be handled otherwise. The equilibrium state in the presence of angular momentum is not uniform but its local temperature varies along the radius of rotation.

2. Thermodynamics with Angular Momentum

It is known from the controversy in 60’s that the local temperature of a relativistic rotating body is not uniform in equilibrium: the outer parts of a rotating body has higher temperature than the inner parts (Stuart & Brainard, 1970; Israel, 1976).

There can be an intuitive explanation to this non-uniform local temperature. Suppose a gas in a gravity field like the atmosphere of the earth. The density of the gas in equilibrium is larger at the bottom because of the gravity force on the particle mass. The same effect will work on the energy, which is equivalent to mass in relativity; the gravitational force act as to attract energy downward, and the energy density will be larger at the bottom. In a rotating system, the centrifugal force plays the role of gravity, and makes the energy density higher at the outer part.

This non-uniform temperature does not fit well to the basic concept of thermodynamics, where a matter in equilibrium should be uniquely determined with a small number of global quantities (energy, temperature, etc.). To overcome this difficulty we can define a global temperature when the body is in equilibrium. The entropy in Equation (1) is then expressed as

\[
dS = \beta \mu dP^\mu + \lambda_{\mu \nu} dL^{\mu \nu},
\]

where \(L^{\mu \nu}\) is the angular momentum tensor and \(\lambda_{\mu \nu}\) is the corresponding intensive parameter, which can be regarded as a generalized inverse temperature for angular momentum. In the above expression, both \(\beta\) and \(\lambda_{\mu \nu}\) are the global parameters, i.e., they are not functions of position but single constant numbers.

One might think it is not necessary to introduce global parameters like \(\beta\) and \(\lambda_{\mu \nu}\) as long as we can calculate the equilibrium state with the local parameters, but it is not true. Global parameters are necessary in interactions with non-local fields. A good example to demonstrate the usefulness of global temperature is a rotating blackbody cavity with photons inside. When we wish to calculate the distribution of photons in equilibrium with the cavity wall, local temperature is not appropriate because photons (i.e., quantized electromagnetic field) have fundamentally non-local nature.

Making use of global parameters enable us to calculate the equilibrium state of photons, which can be calculated as a canonical distribution using the standard procedure of statistical mechanics. The difference is that the angular mo-
momentum conservation should be included as an additional constraint besides the energy-momentum conservation. We have confirmed that the resulting distribution becomes different from the conventional Planck distribution due to the angular momentum effect.

Recently papers have been published on the importance of the orbital angular momentum of electromagnetic waves. Harwit (2003) suggested its possible applications to astrophysics. The measurement on the angular momentum of electromagnetic waves can give further information from the thermodynamical point of view presented in this paper. We can obtain the information on the rotation of the radiation source if observed electromagnetic waves show an equilibrium distribution with angular momentum. For example, we might be able to know the rotation of the whole universe from the angular momentum temperature of the cosmic microwave background, though it might be very small.

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References
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