Numerical methods used in the Lyon-Fedder-Mobarry Global code to model the magnetosphere

John G. Lyon

Department of Physics and Astronomy, Dartmouth College, Hanover NH USA

We discuss the numerics used in the Lyon-Fedder-Mobarry (LFM) global magnetospheric MHD code. The major defining elements of the LFM are a Total Variation Diminishing (TVD) algorithm incorporating high order spatial differencing, the maintenance of $\nabla \cdot \vec{B} = 0$ through the use of a Yee grid, accommodation to the large $\beta$ variations within the magnetosphere, a grid adapted to the problem, and an integral ionospheric model.

1. Introduction

The Lyon-Fedder-Mobarry code has been around now, pretty close to its original form, for about 20 years. In that time the MHD modeling of the magnetosphere has gone from a rather quixotic effort to an indispensable tool for understanding the magnetosphere-ionosphere system. With respectability have come a number of groups who have developed their own codes. Table 1 shows a list of the major global MHD models in use today. The first column shows the originator; the remaining columns provide some information about the basic numerical techniques used. All of these codes implement the ideal MHD equations. Later on, we will discuss the meaning of the various columns. However, without stressing any of the columns, it is apparent that there is a great deal of variation in the details of how these equations are solved.

<table>
<thead>
<tr>
<th>Code</th>
<th>TVD</th>
<th>Order</th>
<th>Mesh</th>
<th>$\nabla \cdot \vec{B}$</th>
<th>Ionosphere</th>
</tr>
</thead>
<tbody>
<tr>
<td>CISM (LFM)</td>
<td>Y</td>
<td>8</td>
<td>Adapted</td>
<td>Y</td>
<td>Y</td>
</tr>
<tr>
<td>Raeder</td>
<td>Y</td>
<td>4</td>
<td>Stretched</td>
<td>Y</td>
<td>Y</td>
</tr>
<tr>
<td>Michigan</td>
<td>Y</td>
<td>2</td>
<td>AMR</td>
<td>?</td>
<td>Y</td>
</tr>
<tr>
<td>Ogino</td>
<td>N</td>
<td>2</td>
<td>Cartesian</td>
<td>Y</td>
<td>N</td>
</tr>
<tr>
<td>Tanaka</td>
<td>Y</td>
<td>2</td>
<td>Spherical?</td>
<td>Elliptic</td>
<td>Y</td>
</tr>
<tr>
<td>Jahunnen</td>
<td>?</td>
<td>1</td>
<td>AMR</td>
<td>?</td>
<td>Y</td>
</tr>
<tr>
<td>Winglee</td>
<td>N</td>
<td>2</td>
<td>Cartesian</td>
<td>Y</td>
<td>N</td>
</tr>
<tr>
<td>MRC</td>
<td>Y</td>
<td>2</td>
<td>Mixed</td>
<td>Y</td>
<td>Y</td>
</tr>
</tbody>
</table>

The LFM is built on the basis of a number of considerations which we will take in turn:

1) High Resolving Power Transport
2) $\nabla \cdot \vec{B} = 0$
3) $\beta$
4) Adapted Grid
5) Integral Ionospheric Model

2. High Resolving Power Transport

The transport used in the LFM is designed to accomplish two goals: to avoid spurious extrema in the calculation and, at the same time, provide as high a resolution as possible. We illustrate by discussing the solution of the linear advection problem:

$$\frac{\partial f}{\partial t} + v \frac{\partial f}{\partial x} = 0$$ (1)

The solution of which is for a profile, $f$, to be carried to the right at a constant rate of speed, $v$. Figure 1 shows the dilemma that numerical solutions face. The top panel shows the analytic solution of moving a square wave 200 cells. The second shows a first order scheme. Order refers in this case mainly to spatial differencing. First order schemes are the only ones that guarantee no spurious extrema. However, the square wave is severely smeared (diffused). Higher order schemes are formally more accurate, but lead to overshoots as seen in the final panel.

Comparison of Advection: The top plot shows the analytic solution to advecting a square wave 200 cells. The next shows a first order solution, and the bottom a 2nd order (Lax-Wendroff solution). The bottom two have the analytic solution overlaid as a dotted line.

The solution is to combine both a high-order and a first-order scheme through the use of a non-linear switch. Such numerical schemes have come to be known as TVD (Total Variation Diminishing). The non-linear switch chooses which numerical technique on the basis of the local solution.
Fig. 2. Performance of TVD Schemes on Square Wave Advection: The plots from top to bottom show the results from a 2nd order, 4th order, and 8th order scheme used to advect a square wave pulse 200 cells. The analytic solution is shown as a dotted line.

Regions where there are very steep gradients (which cause overshoots) cause the first-order algorithm to be used. Elsewhere, the higher order scheme is used for higher accuracy. This actually corresponds to regions where a Taylor series (or similar expansion) makes sense. Strangely enough, the use of very high order algorithms can make the resolution of discontinuous profiles better as seen in Figure 2 where TVD solutions for 2nd, 4th, and 8th order are shown. All of these are much better than the non-TVD solutions in Fig. 1, but clearly there is a large improvement as the order increases. This is related to the fact that a higher order polynomial can actually model a steeper profile without an overshoot.

3. $\nabla \cdot \vec{B} = 0$

As discussed by [Brackbill and Barnes, 1980], if $\nabla \cdot \vec{B}$ is non-zero, the magnetic stress tensor gives rise to non-physical magnetic forces, $\vec{B} \nabla \cdot \vec{B} / 4\pi$ along the field lines. Particularly in low $\beta$ situations, these can have disastrous consequences. It is possible to formulate the MHD equations to allow a finite $\nabla \cdot \vec{B}$ arising from numerical error. Basically one adds the negative of $\nabla \cdot \vec{B}$ terms that should be zero in the momentum equation and in Faraday's law, thus canceling out the harmful effects. This is the approach taken by [Powell et al., 1999]. This leads to a consistent and stable representation of the MHD equations. This is generally required of fully cell-centered TVD schemes, since the numerical divergence and curl operators may not give zero when TVD switches operate. It is possible, however, to maintain $\nabla \cdot \vec{B} = 0$ with TVD switches. In the late 1970’s Klaus Hain, John Lyon and Steve Zalesak all developed more or less independently (even though they were working in the same group at NRL) the key insight: TVD schemes could be used for magnetic field and maintain $\nabla \cdot \vec{B} = 0$ if the so-called Yee [Yee, 1966] grid is used. Lyon’s method was used for 2D calculations [Lyon et al., 1981] and later in the LFM code itself. The technique(s) unfortunately were not documented, but the basic ideas were later rediscovered by Evans and Hawley [Evans and Hawley, 1988] and Devore [Devore, 1991].

Figure 3 shows the Yee type grid used in the LFM. The primary magnetic field quantities are the flux threading the faces. For a Cartesian grid, as shown, $B_x$ is offset half a cell from the cell center in the x-direction. In a non-Cartesian grid, the actual quantity is $\int dA \hat{n} \cdot \vec{B}$, the integral of the normal of the cell face dotted into the magnetic field. This quantity is updated using Stokes theorem with the electric field along the edges of the face.

4. Low $\beta$

In the inner magnetosphere, the generally low $\beta$ of the plasma leads to a situation in which the pressure derived from the total energy equation becomes inaccurate because the energy is strongly dominated by the magnetic energy. This leads to casting the MHD equations in a somewhat unfamiliar form:

$$\frac{\partial \rho}{\partial t} = -\nabla \cdot \rho \vec{v} \tag{2}$$

$$\frac{\partial \rho \vec{v}}{\partial t} = -\nabla \cdot \left( \rho \vec{v} \vec{v} + I P \right) - \nabla \cdot \left( \frac{B^2}{8\pi} - \frac{\vec{B} \cdot \vec{B}}{4\pi} \right) \tag{3}$$

$$\frac{\partial E_P}{\partial t} = -\nabla \cdot \left( \vec{v} \left( \rho \vec{v}^2 / 2 + \frac{\gamma}{\gamma - 1} P \right) \right) - \vec{v} \cdot \nabla \cdot \left( \frac{B^2}{8\pi} - \frac{\vec{B} \cdot \vec{B}}{4\pi} \right) \tag{4}$$

$$\frac{\partial \vec{B}}{\partial t} = \nabla \times (\vec{v} \times \vec{B}) \tag{5}$$

where $E_P = \rho \vec{v}^2 / 2 + \frac{P}{\gamma - 1}$

The major difference between the standard set of conservative ideal MHD equations and the above is the use of an equa-
A Typical Magnetospheric Grid: The inset shows one plane of the full grid, while the larger image shows a blow-up of the region around the Earth.

A boundary condition, but one that may have a profound effect on magnetospheric structure. The inner boundary of the calculation is where the ionosphere is coupled to the magnetosphere. The ionospheric potential is solved using the thin spherical shell approximation

\[ \nabla \cdot \Sigma \cdot \nabla \Psi = j_\parallel \sin \delta \]

\[ \text{where } \Sigma = \begin{pmatrix} \Sigma_P / \sin^2 \delta & -\Sigma_H / \sin \delta \\ \Sigma_H / \sin \delta & \Sigma_P \end{pmatrix} \]

where \( \Sigma \) is the conductivity tensor, \( \delta \) is the dip angle, and \( \Psi \) is the electrostatic potential. This equation is solved on an ionospheric grid built by mapping the inner boundary of the MHD calculation along dipole field lines to the ionosphere. The filed aligned currents are calculated at the inner boundary by evaluating \( \nabla \times \vec{B} \cdot \vec{B} / B^2 \), which can be mapped directly to the ionosphere. The conductances needed for the solution contain both a solar EUV and a precipitation component. The calculation of the conductances used has been discussed in detail in Fedder [Fedder et al., 1995], and will not be discussed further here.

Once the electric potential is calculated, the potential itself is mapped out to the inner boundary of the MHD calculation where it provides a boundary condition for Faraday’s Law and for the \( \vec{E} \times \vec{B} \) drift.

7. How much does all this mean?

The table at the beginning of this abstract shows that there is a fair divergence in solution techniques for the practitioners of global MHD. There is enough difference that there can often be substantial differences in the solution coming out of different codes for the same exact problem.

Figure 5 shows a rather extreme example. The two panels show the results at the same time for a simulation of a Northward IMF hitting the magnetosphere. The solar wind parameters are \( n = 5/\text{cm}^3 \), \( v = 400 \text{ km/s} \), and \( B = 5 \text{ nT} \). Both panels result from using the LFM code, but in the top panel a purely first order scheme is used, while the bottom the usual high-order TVD technique employed. The colored plane is noon-midnight with the color indicating the \( v_x \) component. Unit vectors of magnetic field are also displayed. The sun is off to the right. Note that through most of the magnetosphere there is tailward flow in the diffusive case, even around the x-line above the cusps shown by the magnetic field vectors. On the other hand, the lower diffusion bottom panel shows that, for this Northward case, there is very little flow within the magnetosphere and the usual jetting out of the x-line is observed.

The bottom panel seems closer to the observations, but both codes are solving the ideal MHD equations to some degree of accuracy. There are probably two lessons to bear in mind: first, comparison against observation is indispensable in evaluating a code result and, second, it’s always good to have some idea of the numerical limitations of a given code.

Afterword: A detailed discussion of the numerical techniques in the LFM can be found in [Lyon et al., 2004].
Fig. 5.
Difference between a diffusive (top) and (non-diffusive) solution for the magnetosphere during Northward IMF. Shown is the non-midnight meridional plane with the \( x \)-velocity shown as false color. The unit magnetic field vectors are shown in the

References


