Vlasov Code Applications

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In this paper we discuss Vlasov codes and Vlasov simulations. Due to their excellent resolution, they have proven to be a useful tool to study the non linear resonant wave-particle interaction. The goal of the paper is to give some insight to the question of choosing between PIC and Vlasov codes, both from a numerical and a physical point of view. An application to gyrokinetic modelling of strongly magnetised plasmas will be given.

1. Introduction

Vlasov models have long been used to study collisionless plasmas. Although very interesting analytical results are obtained from more simple fluid models, the importance of resonant wave-particle interaction points to the use of the fully kinetic Vlasov equation and furthermore to the use of numerical simulation. However, these simulations, employing as they do the well-known Particle-In-Cell (PIC) technique, have difficulty in supplying a usefully precise description of the electron acceleration process. This is because the PIC codes lack enough particles to display the detailed phase-space structure of the distribution function which is often obtained in those regions of phase space where particle and phase velocities are comparable and where trapping occurs.

On the other hand direct solution of the Vlasov equation itself on a phase space grid (the so called Vlasov codes) is a powerful tool for studying in details the particle dynamics due to the very fine resolution in phase space. Since supercomputers have an exponential increase both in power and memory, Vlasov codes are becoming more and more popular.

The goal of this paper is to help the Reader to select problems for which Vlasov models and Vlasov simulations could be pertinent both from a numerical (CPU time, memory size) and physical point of view. Some pedagogical examples will be presented and discussed.

2. Vlasov plasmas

Before introducing the principles of Vlasov codes it is worth to briefly recall when and under what conditions the Vlasov equation is valid. This is a complex question.

In the Vlasov equation the particles interaction is entirely described by a “mean” field computed via the Poisson equation or more generally via Maxwell’s equations for more complex plasmas. It means that we stay at a space scale where fluctuations can be ignored, although the plasma is a complex plasma. It means that we stay at a space scale described by a “mean” field computed via the Poisson equation (and furthermore to the use of numerical simulation). Before introducing the principles of Vlasov codes it is worth to briefly recall when and under what conditions the Vlasov equation (sometimes called collisionless Vlasov equation) is valid for plasmas with \( n_0 \lambda_D^3 \gg 1 \) and for a time shorter than \( n_0 \lambda_D^3 \omega_p^{-1} \).

To be more precise, we follow an idea originally introduced by N. Rostoker and M. Rosenbluth (1960) and developed by Lotte and Feix (1984), which is based on a ”virtual” dichotomy experiment where each particle \((e,m)\) is divided into two subparticles with for each a charge \( e/2 \), a mass \( m/2 \) and a kinetic energy divided by two (and consequently the same velocity). Let us repeat this dichotomy \((D)\).

Now we look for the length and time scale invariant under \((D)\) and a parameter \( g \) which is divided by two under \((D)\). It can easily be shown that \( \lambda_D \) and \( \omega_p^{-1} \) are the length and time scale invariant under \((D)\) and we immediately get

\[
g = \frac{1}{n_0 \lambda_D^3} \tag{1}
\]

Therefore the Vlasov equation is obtained in the limit \( g \rightarrow 0 \) and is the sole kinetic equation which remains invariant under \((D)\).

In many cases the Vlasov equation is an excellent approximation and remains valid up to time of order \((\omega_p g)^{-1}\): fusion plasmas exhibit a ”graininess” factor \( g \) of order \( 10^{-7} \) to \( 10^{-6} \) and space plasma can have still smaller \( g \).

3. Vlasov codes

Plasma codes have now a long history since the first calculations have been performed in the late 50s using a few hundred macroparticles moving only in a 1D plasma (for an extensive bibliography on the subject, see for instance the book by Birdsall and Langdon, 1991).

Since collective phenomena are our concern, it is worth to remember that Coulomb interaction between charged particles are actually replaced in a Vlasov plasma by a mean field calculated from Poisson or Maxwell equations using charge density or current density where the microscopic fluctuations (due to the fact that a plasma is not a continuum) are averaged over the Debye length \( \lambda_D \).

This mean field concept is the basic idea of Particle in Cell (or PIC) codes: during a time step, particle trajectories are computed from a field prescribed on a fixed grid with a mesh size of order \( \lambda_D \), which is equivalent to compute the characteristics of the Vlasov equation; at the end of the time step the charge of each particle is redistributed among the neighboring mesh points allowing to solve Poisson’s (or
Maxwell’s) equation from grid data. This method yields satisfying results with a relatively small number of particles. But the numerical noise due to the individual effects may be too important to get an accurate description of the distribution function.

Therefore it was early realised that a numerical solution of the Vlasov equation itself could be more appropriate and G. Knorr (1963) gave the first solution through a spectral method (a double Fourier transform both in \(x\) and \(v\)) but he considered only weakly non linear problems close to the linear theory.

Finally an accurate treatment of the 1D Vlasov-Poisson equation was given by C.Z. Cheng and G. Knorr (1976). The method is the basis for modern Vlasov codes. It introduces two ideas:

- The time splitting: it consists in separating the free streaming part \(\partial_t f + v \partial_x f = 0\) from the acceleration part \(\partial_t f + (e/m)E(x,t)\partial_x f = 0\); therefore the integration is divided into four steps: let \(f^n\) be the distribution function at time \(n\Delta t:\)
  1) compute \(f^*(x,v) = f^n(x - v\Delta t/2, v)\);
  2) compute the electric field \(E^*\) by substituting \(f^*\) into the Poisson equation;
  3) compute \(f^{**}(x,v) = f^*(x,v - eE^*/m\Delta t)\);
  4) repeat again step 1, giving \(f^{n+1}(x,v) = f^{**}(x - v\Delta t/2, v)\).

- The reconstruction of \(f^*\), \(f^{**}\) and \(f^{n+1}\) on a mesh in phase space: to do that, cubic spline interpolations, Fourier interpolation, as well as finite volume methods can be used. For the Vlasov-Poisson system, this scheme is equivalent to an integration of the Vlasov equation along the characteristics and is correct up to the second order.

During the 80s PIC codes were able to perform 2D simulations getting useful information out of them and direct Vlasov solvers lost their interest for most people due to their high numerical cost. But now very powerful (as well in CPU performance as in memory size) parallel or vector computers are available and the low noise high resolution Vlasov codes are getting more and more popular. They have been extended to relativistic laser-plasma or gyrokinetic simulations. The original algorithm of Cheng and Knorr (1976) has been cast into the more general framework of semi-Lagrangian methods by E. Sonnendrucker, J. Roche, P. Bertrand and A. Ghizzo (1999) and T. Nakamura and T. Yabe (1999).

4. PIC versus Vlasov codes

Solving Poisson’equation (or Maxwell’s equations) is the same task for both PIC and Vlasov codes and need the same spatial grid, with a mesh size \(\Delta x\) of the order of the Debye length \(\lambda_d\). The number of mesh points of the spatial grid for both codes can be estimated as \(N_x^d = (L/\Delta x)^d\) for a plasma with a typical length \(L\), \(d_x\) being the spatial dimension of the problem (\(d_x = 1, 2\) or 3).

On the other hand step 1 and 3 of the Cheng-Knorr scheme for the Vlasov code described above, are nothing else but solving the characteristic equations for one particle. This is completely equivalent to solve the motion equations for one particle in a PIC code. Therefore it is clear that pushing one particle in a PIC code needs the same numerical effort than reconstructing a phase space mesh point in an eulerian Vlasov code.

Consequently the ratio between the numerical effort for a PIC code and a Vlasov code (CPU time as well as memory requirement) will scale as \(N_{vlas}/N_{part}\) where \(N_{vlas}\) is the total number of mesh points in phase space for the Vlasov equation and \(N_{part}\) is the total number of superparticles in the corresponding PIC code.

For a Vlasov code \(N_{vlas}\) can be written simply as the product of the spatial grid by the velocity grid

\[
N_{vlas} = \left( \frac{L}{\Delta x} \right)^d \frac{N_v}{d_v} \quad (2)
\]

where \(d_v\) is the dimension of the velocity space (\(d_v = 1, 2\) or 3). Usually \(d_v = d_x\) but in some cases especially when the characteristics of the Vlasov equation possess an exact invariant the two numbers may be different.

For a PIC code (where \(n_0\) is the density of superparticles)

\[
N_{part} = n_0 L^d_{dx} = n_0 (\Delta x)^d \left( \frac{L}{\Delta x} \right)^d \quad (3)
\]

In (3) we see the appearance of \(n_0\Delta x^d\) which is nothing else but the inverse of the graininess parameter (see eq. 1) due to particle discreteness of the PIC code

\[
g_{pic} = \frac{1}{n_0 \lambda_D^d} \quad (4)
\]

This relation deserves a few comments. A PIC code exhibits a paradoxal situation where the introduction of a spatial grid \(\Delta x \sim \lambda_D\) allows the description of the collective effects through the mean field approximation but at the price of the reintroduction of individual effects due to the finite number of superparticles. These individual effects are often considered as a numerical noise whose measure is clearly \(g_{pic}\) as given by (4). To keep the coherence of the superparticle model we must have \(g_{pic}\) as small as possible. \(N_{part}\) can be now written

\[
N_{part} = g_{pic}^{-1} \left( \frac{L}{\Delta x} \right)^d \quad (5)
\]

Finally from (2) and (5) we get the ratio

\[
\frac{N_{vlas}}{N_{part}} = g_{pic}^{-1} N_v^d \quad (6)
\]

To illustrate this relation consider a typical value \(N_v = 100\) which is quite reasonable to have a fine sampling of the velocity space. From (6) we can draw table 1.

Although these numbers must be considered with caution they give the general ideas for a choice between PIC or
5. Resonant wave-particle interaction in phase units

\[ \omega - k x \]

simulate this process (see for instance T. W. Johnston, P.

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tering process, long of interest for laser fusion. As is well

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years. One of the potentially most effective ways to pro-

plasma accelerators. The acceleration of charged particles

\( \sim \frac{\omega}{k c} \approx 1 \) large-amplitude longitudinal electron plasma wave has received considerable attention in recent years. One of the potentially most effective ways to pro-

duce such a wave uses the forward Stimulated Raman Scat-

tering process, long of interest for laser fusion. As is well

known, this is a stimulated decay of incident laser light into a forward-scattered light wave and a forward-going plasma wave (necessarily at plasma densities below the pump fre-

quency quarter-critical value). Vlasov simulations have been performed using a 1D relativistic Vlasov–Maxwell code to simulate this process (see for instance T. W. Johnston, P.

Bertrand et al., 1992).

Fig. 1 shows the very accurate phase space representation of layering and folding seen in phase space due to electron trapping, detrapping, and retrapping. These details would have been invisible in the usual particle-in-cell (PIC) code because of graininess at low phase space densities.

### Table 1. PIC code vs. Vlasov code: ratio \( N_{\text{vlas}}/N_{\text{part}} \)

<table>
<thead>
<tr>
<th>( g_{\text{pic}} )</th>
<th>( d_x = 1 )</th>
<th>( d_x = 2 )</th>
<th>( d_x = 3 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \times 10^{-2} )</td>
<td>1</td>
<td>( 10^2 )</td>
<td>( 10^4 )</td>
</tr>
<tr>
<td>( \times 10^{-4} )</td>
<td>( 10^{-2} )</td>
<td>1</td>
<td>( 10^2 )</td>
</tr>
<tr>
<td>( \times 10^{-6} )</td>
<td>( 10^{-4} )</td>
<td>( 10^{-2} )</td>
<td>1</td>
</tr>
</tbody>
</table>

Vlasov code. Below the principal diagonal Vlasov codes offers a better resolution at a smaller price, while, above the diagonal, PIC codes must be preferred. The crucial point is the dimension of velocity space:

- **For 1D velocity space** which is the case for gyrokinetic modelling (see part 6) Vlasov codes must be preferred.

- **For 2D velocity space** the question is open and depends on the level of noise which is bearable in the PIC code; if a small level is needed to study the onset of instabilities (with \( g_{\text{pic}} \) smaller than \( 10^{-4} \) for instance) a Vlasov code has to be used. On the contrary, for gross phenomenon (where a small \( g_{\text{pic}} \) is not needed) a PIC code will do the job at a lower price.

- **For 3D velocity space** unless the need of ultra low noise with very small \( g_{\text{pic}} \). PIC codes remain the only choice.

6. Application to gyrokinetic modelling

Low frequency ion-temperature-gradient-driven (ITG) instabilities are now commonly held responsible for turbulence giving rise to anomalous radial energy transport in the core of tokamaks. The computation of turbulent thermal diffusivities in fusion plasmas is of prime importance since the energy confinement time is determined by these transport coefficients. During recent years, ion turbulence in tokamaks has been intensively studied both with fluid (see for instance X. Garbet and R.E. Waltz, 1996) and kinetic simulations (see R.D. Sydora, V.K. Decyk and J.M. Dawson, 1996, using PIC simulations, and G. Depret, X. Garbet, P. Bertrand and A. Ghizzo, 2000, using Vlasov simulations). Although more accurate, the kinetic calculation of turbulent transport is much more demanding in computer resources than fluid simulations. As a matter of fact the thermal diffusivity \( \chi \) computed from fluid simulations exhibit an overestimate as compared to kinetic simulations (with a fluid prediction twice the kinetic prediction i.e. \( \chi_{\text{fluid}} \approx 2 \chi_{\text{kinetic}} \)).

In principle, one has to solve a 6D kinetic equation to determine the distribution function but for strongly magnetized plasmas, averaging the kinetic equation over the cyclotron motion, which is faster than turbulent motion, reduces the dimensionality. The physical model is based on the Vlasov gyrokinetic equation for the ions with an adiabatic response for the electrons. It can be easily shown that for a low frequency turbulence, a gyrokinetic equation can be used, which corresponds to a gyro-average over the cyclotron motion (see T.S. Hahm, 1988). We introduce the guiding centre ion distribution function \( f_\mu (\vec{r}, \vec{v}_\parallel, t) \) where \( \vec{r}, \vec{v}_\parallel, \mu \) are respectively the position, the parallel component of the velocity along \( \vec{B}_0 \) and the first adiabatic invariant \( \mu = v_\parallel^2 / 2B_0 \) of the ion guiding centres. The time evolution is given by the gyro-averaged Vlasov equation written here for a uniform \( \vec{B}_0 \)

\[
\frac{\partial f_\mu}{\partial t} + \left( \vec{v}_\parallel \cdot \vec{B}_0 + \vec{E} \times \vec{B}_0 / B_0^2 \right) \cdot \nabla f_\mu + \frac{q_i}{m_i} \langle E_\parallel \rangle \frac{\partial f_\mu}{\partial \vec{v}_\parallel} = 0 \tag{7}
\]
where $\vec{b}$ is the unit vector along $\vec{B}_0$. $\bar{\phi}$ means gyro-averaging over the Larmor radius $\rho_L$.

Assuming adiabatic electrons and using the long wavelength limit Poisson’s equation is replaced by the quasi-neutrality condition

$$n_0 \left( 1 + \frac{e}{T_e} (\phi - \bar{\phi}) \right) = \langle n_i \rangle + \frac{1}{B_0 \Omega_{ci}} \nabla \cdot (n_0 \nabla \phi) \quad (8)$$

where $\bar{\phi}$ is the potential averaged over a magnetic surface and $\langle n_i \rangle$ denotes the gyro-averaged ion density. In eq. (8) $n_0(r)$ and $T_e(r)$ are the initial density and electron temperature profiles.

As a first remark we must notice that $\mu$ is actually a continuous label in eq. (7). Only a finite number of discrete values of $\mu$ can be considered thus decreasing drastically the computational effort. Even one single value can be chosen. For a given choice of $\mu$ we define a given distribution function $f_\mu$ and the different $f_\mu$ are coupled through the gyro-averaged quasi-neutrality equation (8).

This model is obtained within the frame of the usual gyrokinetic ordering:

$$\omega/L_{ci} \sim k_{\parallel}/k_{\perp} \sim e\phi/T_e \sim \mu L_{grad} \sim \varepsilon$$

where $L_{grad}$ is the length of the gradient scale (both density and temperature gradient) in the perpendicular direction and $\varepsilon$ is a small parameter ($\varepsilon \sim 10^{-3}$ in a tokamak plasma). Nevertheless this gyrokinetic approximation is known to preserve the Hamiltonian structure of the original Vlasov-Poisson model even in non uniform magnetic field $\vec{B}_0$ (like an actual tokamak magnetic configuration).

But the most important and interesting feature of (7) is that $f$ depends only on the velocity component $v_\parallel$ parallel to $\vec{B}_0$.

As noticed above in table 1, we are in the most favorable case with $d_\varepsilon = 1$ which points to the use of a Vlasov code instead of PIC.

The cylinder ITG (Ion Temperature Gradient) instabilities correspond to small scale instabilities, which grow and saturate to a state of developed turbulence during the non linear phase of the instability. A new gyrokinetic Vlasov code has been developed in a cylinder geometry corresponding to a periodic cylindrical plasma which can be considered as a limit case of a stretched torus. See the paper by V. Grandgirard, M. Brunetti, P. Bertrand et al. (2005). This code permits to simulate turbulence well into the nonlinear regime with no numerical disruption and a good conservation of the total energy with an error smaller than 2%. As an example the distribution function during the fully developed turbulent phase in a $(r, \theta)$ cross section is represented on fig. 2.

7. Conclusion

The numerical simulation is a huge topic with hundreds of papers published every year. Vlasov simulations are slowly introduced in place of the lagrangian PIC models for two reasons: the lack of numerical noise and the good resolution in phase space, provided the dimension of velocity space is as low as possible. The very good phase space resolution allows a precise study of wave-particle interaction including acceleration and trapping. Kinetic effects in plasmas allow more phenomena than are found using only fluid theory with ad hoc kinetic damping.

Finally huge progress is expected from new adaptive time dependant grids with small mesh size in the non zero regions and large mesh size for the sparse regions.

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References


