Unstructured Grids and Finite Elements in Space Plasmas Modelling

R. Marchand, J.Y. Lu, K. Kabin R. Rankin, and T. Keeler

Department of Physics, University of Alberta, Edmonton AB, T6G 2J1, Canada

Finite elements with unstructured grids are common in the field of engineering, but they are seldom used to model space plasmas. Yet, this technique offers unique advantages that nicely complement other approaches. In particular, it lends itself naturally to local refinement and adaptivity, and it offers the possibility of working with meshes that are aligned along preferred directions in space. This aspect is particularly important when modelling strongly anisotropic processes, such as energy transport or the propagation of shear waves along the magnetic field. In these cases, transport is predominantly along the magnetic fields, and it is essential to use a mesh that is aligned along the magnetic field lines in order to avoid spurious numerical diffusion in the perpendicular direction.

1. Introduction

Today’s accepted Global Circulation models used to simulate the Earth magnetosphere and its coupling with the solar wind are largely based on the finite volume or spectral discretization techniques. While numerically very efficient and capable of accounting for a rich set of physical processes, these approaches tend to make use of meshes that have a fixed orientation in space, or of spatial representations that cannot be oriented along the preferred direction of the physical system. This can be a serious drawback when modelling plasmas characterized by a strong anisotropy. As an example, thermal energy transport can be many orders of magnitude larger in the direction parallel to the magnetic field, than in the perpendicular direction. The propagation of certain disturbances (shear Alfvén or whistler waves for example) can also be strongly anisotropic. In those cases, the use of a mesh with fixed orientation in space would lead to unacceptable spurious transport in the direction perpendicular to the field lines, and it could not be used to model such phenomena.

In those cases, the method of finite elements with unstructured triangular (2D) and tetrahedral (3D) meshes offers an interesting alternative. Because of the possibility of locally orienting cells of an unstructured grid along the magnetic field, it is then possible to capture strong anisotropic processes while working with meshes of acceptable sizes. In this paper, we briefly present the finite element method, together with an approach to generate aligned meshes in two and three dimensional geometries. We then show example results illustrating simulations of shear Alfvén waves resonances in Earth magnetosphere.

2. Finite element discretization and the generation of field aligned unstructured meshes

The magnetosphere can be simulated in the MHD approximation. It is then described by a set of coupled partial differential equations for eight dependent variables (density, pressure and the three components of the magnetic field and fluid velocity vectors) with initial and boundary conditions. These equations, in turn, can be discretized with finite elements by expressing each dependent variable, as a superposition of interpolating functions with compact support on an unstructured triangular (2D) or tetrahedral (3D) grid. This produces a large set of equations described by a sparse matrix. Solutions can be obtained either directly (for the smaller 2D problems) or iteratively. In our model, we typically solve all equations simultaneously and fully implicitly. For large systems of equations, however, it is possible to use an operator splitting technique where the set of equations is broken into groups that are solved for separately.

An essential aspect of mesh generation is its alignment along the magnetic field. This is accomplished by distributing mesh vertices along selected magnetic fields and connecting them so as to form constrained Delaunay cells (or elements). In practice, distributing mesh vertices along field lines so as to form a quasi-orthogonal connection across field lines is found to give good quality meshes (where the constrain only has a minor impact).

3. Simulations in two dimensions

Our 2D model has already been used to study aspects of shear Alfvén wave resonances in the magnetosphere (J.Y. Lu, et al.)

![Relative perturbed density profile computed for a shear Alfvén wave resonance localizing at L=7.7 in dipole geometry.](image)
2003). The 2D approximation assumes near azimuthal symmetry and, strictly speaking, it is only valid in along low L shells where the magnetic field is nearly dipolar. It is otherwise fully nonlinear, it accounts for the coupling between shear and compressional modes, and for non-linear feedback provided by ionospheric heating. Resonant shear waves can be driven by mode conversion of compressional waves in the magnetosphere. For simplicity, the modes are excited by driving the toroidal plasma velocity at a fixed frequency over a broad area. After several periods, the wave phase mixes and localizes to a narrow layer near the resonant surface. The boundary conditions imposed at the ionosphere correspond to an initial conductance of 2 S. The ionospheric conductance is calculated self consistently from the computed parallel current and a semi-empirical model of ionization balance (Lu, et al. 2005). In this case, the lowest eigenmode has a symmetric velocity profile and an antisymmetric perturbed toroidal magnetic field profile, with respect to the equatorial plane. As the mode localizes and its amplitude increases, the associated ponderomotive force “pushes” plasma along the magnetic field, away from the region near the ionosphere toward the equator. Earlier calculations of this process (Lu, et al. 2005) were done in the linear approximation, with the only nonlinear dynamics coming from the feedback provided by ionospheric heating. The simulation results reported here account for the full wave nonlinearities, including its coupling with compressional waves.

The resulting relative perturbed density calculated near saturation is illustrated in Fig. 1. The parallel current induced by the shear wave is also responsible for heating the ionosphere, and for increasing its conductance. This, in turn, provides a nonlinear feedback mechanism on the wave that tends to localize it further spatially. Figure 2 shows the computed Pedersen conductance at the same time as in Fig. 1.

Test simulations were made to reproduce, in 3D, results obtained with our 2D model in situations where the 2D approximation is valid (i.e., with azimuthal symmetry). Meshes have also been generated with elements that are aligned along the magnetic field under more realistic conditions of the magnetosphere. Specifically, these were generated for numerically computed states of the magnetosphere obtained with the BATS-R-US global MHD code (Powell, et al. 1999).

Figure 3, illustrates such a mesh computed for a small flux bundle that originates from the northern ionosphere with a circular cross section. The bending and stretching of the magnetic field lines caused by the interaction with the solar wind are clearly visible. Simulation meshes in confined regions such as this can be used to study the dispersion properties of field resonances or the propagation of pulses (shear Alfvén or whistler) along the magnetic field. Meshes covering larger simulation domains are also being generated to study more global aspects of the magnetosphere dynamics, including the conversion of compressional waves into shear modes.

4. Extension to three dimensions

Our finite element code has been extended to three dimensions. The procedure here is conceptually the same as with the 2D approach except that, computationally, the problem is much more challenging. The finite element discretization results in considerably more equations that need to be solved simultaneously and, in practice, these can only be solved iteratively. In our code, we have adopted Saad’s GMRES iterative solution technique with threshold LU preconditioning (Saad, 2003). Also, we note that the generation of good quality meshes is not as straightforward as in two dimensions and, in order to ensure mesh alignment, we had to develop our own unstructured tetrahedral mesh generator.

Fig. 2: Parallel current density and conductance profiles across the magnetic field lines near the ionosphere

Fig. 3: Illustration of a magnetic flux tube in which an unstructured field aligned 3D mesh is constructed

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References