Generation of Phase Coherence Among Foreshock MHD Waves: Data Analysis and Multiple-Triplet Model

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Large amplitude MHD waves observed in the earth’s foreshock are not completely phase random (as assumed in quasi-linear theories), but are almost always phase correlated to a certain degree. Furthermore, the larger the MHD wave amplitude is, the stronger is the phases correlation, implying that the detected phase coherence is a consequence of nonlinear interaction among the MHD waves. In order to understand physical processes leading to the generation of the phase coherence, we construct a simple model of weak turbulence consisting of multiple triplets, which are the minimum units of nonlinear interaction among eigenmodes. Long time statistical behavior of the multiple-triplet system depends on how the triplets are connected: if the connections are given in such a physical way that the frequency of each mode can be uniquely determined, the system approaches the self-organized critical state as time elapses; otherwise, the system does not evolve to any statistically interesting state. Adjacent modes often tend to phase synchronize as the modes exchange energy between them. Implications of the results to the phase coherence found in the foreshock MHD waves are discussed.

1. Introduction

Magnetohydrodynamic (MHD) waves are ubiquitous in space plasma. In particular, those found in the earth’s foreshock region are characterized with order of unity normalized magnetic field amplitude and extremely unique waveforms (such as the so-called shocklets), indicating that nonlinear processes play major roles in generating these waves. From this perspective, the foreshock MHD turbulence may be regarded as a natural ideal example from which one can learn much on complex dynamical systems description of the space plasma.

In this paper we wish to emphasize the importance of phase distribution in the MHD turbulence. When we Fourier analyze time series data of turbulence, usually much more attention is paid to the power spectrum of the signal than to the phase distribution, although these two contain equal amount of information. Furthermore, the latter reflects the nonlinear coupling between the eigenmodes (i.e., the phase synchronization). Nevertheless, it is often assumed that the turbulence found in space is a random phase mixture of many MHD waves. Furthermore, the random phase approximation has been employed as one of the fundamental assumptions in quasi-linear theories of particle transport.

We introduce a method to quantitatively evaluate the phase coherence of a given time series data, by comparing structure functions of original data and its phase-shuffled and phase-equalized surrogates. Then we discuss that the foreshock MHD waves are not completely phase random, but are almost always phase correlated to a certain degree. We also argue that the larger the turbulence level is, the stronger is the phases correlation, implying that the detected phase coherence is a consequence of nonlinear interaction among the MHD waves.

In order to understand physical processes leading to the generation of the phase coherence, we examine a simple model of the turbulence consisting of multiple triplets, which are the minimum units of nonlinear interaction among eigenmodes. Long time statistical behavior of the multiple-triplet system critically depends on how the triplets are connected: if the connections are given in such a physical way that the frequency of each mode can be uniquely determined, the system approaches the self-organized critical state as time elapses; otherwise, the system does not evolve to any statistically interesting state. Adjacent modes often tend to phase synchronize as the modes exchange energy between them. Implications of the results to the phase coherence found in the foreshock turbulence will be discussed.

2. Observation of the phase coherence

We have recently developed a method to quantitatively evaluate the phase coherence among waves from a given time series data, by comparing structure functions of original data and its phase-shuffled surrogates (Hada et al., 2003; Koga and Hada, 2003). Here we describe the method briefly.

Suppose we have a time series data (measured by experiment or generated by numerical simulation) representing the turbulence, for example, one component of the magnetic field, \( b(t) \). The Fourier transformation of \( b(t) \) gives the power spectrum, \( P(\omega) \), and the wave phase distribution, \( \phi(\omega) \). A simple inspection of \( \phi(\omega) \) does not reveal meaningful information, since the wave phase depends on the choice of the coordinate origin, which is arbitrary, and thus the wave phases almost always appear to be randomly distributed in the Fourier space. In order to avoid this situation, we evaluate the phase coherence in the real space instead of the Fourier space. From the original data (ORG), we make two surrogates, by randomly shuffling the wave phases (the phase randomized surrogate, or the PRS), and by making all the phases equal (the phase correlated surrogate, PCS). The power spectrum of the surrogates are made equal to that of the original data. The differences of these datasets can be

\[
\begin{align*}
\Delta P(\omega) &= P_{\text{PRS}}(\omega) - P_{\text{ORG}}(\omega), \\
\Delta \phi(\omega) &= \phi_{\text{PRS}}(\omega) - \phi_{\text{ORG}}(\omega), \\
\end{align*}
\]

where \( P_{\text{PRS}}(\omega) \) and \( \phi_{\text{PRS}}(\omega) \) are the power spectrum and phase distribution of the phase randomized surrogate, respectively. The phase coherence is defined as the amount of information provided by the wave phases and is given by

\[
\frac{\Delta \phi(\omega)}{\Delta P(\omega)}
\]

In this paper we introduce a method to quantify phase coherence, by comparing structure functions of original data and its phase-shuffled and phase-equalized surrogates.
captured by the structure function,

\[ S(m, \tau) = \sum_{t} |b(t + \tau) - b(t)|^m \]  

where the time lag \( \tau \) characterizes the magnification level of the data.

\[ S(1, \tau) \]

Figure 1 shows \( S(1, \tau) \) for the OBS, PRS, and the PCS datasets. Since we chose \( m = 1 \), the structure function essentially gives the path length measure by the unit \( \tau \). Although the differences are small, \( S(1, \tau) \) for the three datasets are different, indicating that waves contained in the original data are phase correlated to a certain degree. We define the phase coherence index,

\[ C_\phi = \frac{S_{PRS} - S_{ORG}}{S_{PRS} - S_{PCS}} \]  

where \( S_1 \) denotes the value of \( S(1, \tau) \) for dataset *. If the original data is random phase, \( C_\phi \) should be near zero, while \( C_\phi \) if the waves are almost completely phase correlated.

Figure 2 shows \( C_\phi \) evaluated using Geotail magnetic field data, versus normalized magnetic field power amplitude. The positive correlation in the plot implies that the detected phase coherence is a consequence of nonlinear interaction among the MHD waves, which is in agreement with the well-known result in the study of multiple-degrees of freedom dynamical systems that the phase synchronization is a consequence of nonlinear interaction between eigenmodes (waves).

By evaluating \( C_\phi \) for frequency filtered data, one can identify the waves responsible for the generation of the phase coherence. For the foreshock MHD turbulence, it is concluded that such waves lie within the frequency range of \( \sim 0.1 \Omega_i \) to \( \sim \Omega_i \), where \( \Omega_i \) is the ion cyclotron frequency. Based on this result, it is inferred that the MHD waves are first generated around \( \sim 0.1 \Omega_i \) via em ion beam instabilities, as as they are convected by the solar wind, they grow nonlinearly keeping their phases coherent, as the wave energy cascades toward \( \Omega_i \). The phase coherence seems to be lost for the cascade beyond \( \Omega_i \).

![Fig. 1. S(1, \tau) evaluated for the original data (ORG), phase randomized surrogate (PRS), and the phase correlated surrogate (PCS).](image)

3. Multiple-triplet model

In order to understand physical mechanism leading to the generation of the observed phase coherence, we consider a simple model of weak turbulence consisting of many triplets. The triplet is a minimum unit of nonlinear interaction among eigenmodes (waves) that satisfy the resonance condition:

\[ \omega_1 + \omega_2 = \omega_3. \]  

Time evolution of each mode is given by,

\[ \dot{C}_1 = -iC_2C_3^* \]  

\[ \dot{C}_2 = -iC_1C_3^* \]  

\[ \dot{C}_3 = -iC_1C_2 \]  

where \( C_j \) is the complex amplitude of the \( j \)th mode, and the dots represent time derivative. All the coupling coefficients in the above equations are made equal to unity through scaling of the variables. This formalism can be applied to a wide variety of phenomena, including the decay instability of parallel propagating Alfvén waves (Sagdeev and Galeev, 1969).

By defining wave action (number of quanta), \( N_j = |C_j|^2 = E_j/\omega_j \), where \( E_j \) is the energy associated with the mode \( j \), we have Manley-Rowe relations,

\[ N_1 - N_2 = const , \quad N_1 + N_3 = const \]  

Further, if we write \( C_j = A_j \exp(i\phi_j) \), where \( A_j \) and \( \phi_j \) are real, we obtain another invariant,

\[ A_1A_2A_3 \cos \theta = const \]  

where \( \theta = \phi_3 - \phi_2 - \phi_1 \) is the phase difference among the triplet. The triplet set of equations (4-6) is integrable with the
use of eq.(7) and (8). The exchange (‘flow’) of wave action among the triplet is related to the phase difference by

\[ F \equiv \dot{N}_1 = \dot{N}_2 = -\dot{N}_3 = A_1 A_2 A_3 \sin \theta. \quad (9) \]

Now we examine statistical behavior of a multiple degrees of freedom model in which many triplets are connected. Although there are infinitely many ways to couple the triplets, it turns out that statistically interesting states only appear when the coupling is given in such a way that the frequency relations of each mode can be uniquely determined. Here we consider the case where the triplets are sequentially coupled,

\[ \omega_j + \omega_{j+1} = \omega_{j+2} \quad (10) \]

where \( j = 1 \ldots N \) is the mode (site) number, and \( N \) is the total number of the modes. The equation above gives a Fibonacci sequence for the frequency, i.e., \( \omega_j \sim ((1 + \sqrt{5})/2)^j \). The multiple-triplet system still possesses the Manley-Rowe invariants, but is no longer integrable.

The triplets can be written as \( \dot{N}_j = F_{j+2} + F_{j+1} - F_j \), the ‘flow’ and the relative phase are related as

\[ F_j = 2 A_j A_{j-1} A_{j-2} \sin \theta_j. \quad (11) \]

Figure 4 shows numerical result of the relationship between \( F_{19} \) and \( \theta_{19} \), evaluated when the system is at statistical equilibrium. Majority of the data points can be found in two distinct regions: those near \( \theta_{19} \sim 0 \) and \( F_{19} \neq 0 \) correspond to the case where the relative phase is fixed while the mode exchange wave actions, and those near \( \theta_{19} \neq 0 \) and \( F_{19} \sim 0 \) represent the case where the relative phase slips freely when there is no exchange of wave actions among the modes.

4. Discussions

We have discussed in this paper that large amplitude MHD turbulence in space can have finite phase coherence. The phase coherence index, \( C_\phi \), defined using the original as well as phase shuffled and phase correlated surrogates, is evaluated for the foreshock MHD turbulence, and it is found that it is positively correlated with the turbulence level, and that \( C_\phi \) ranges from 0.1 to 0.4 typically, but it reaches to 0.6 sometimes. The simple model of multiply coupled triplet suggests that the phase coherence is naturally generated as the wave action (or the wave energy) is exchanged between the wave modes. In the ‘phase coherent turbulence’ (as opposed to the conventional phase random turbulence), transport of energetic particles can be quite different from our conventional view. Individual motion of particles can be non-Brownian, and may be represented as combination of ‘stick’ (trapped by em field islands) and ‘walk’ (ballistic motion between the em islands) motions. Ensemble of these particles cannot be described as a classical diffusion process, but both sub- and super-diffusion may appear. If this is the case, one is required to invoke the fractal Fokker-Planck formalism.

References